New boundary conditions for integrable lattices

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Abstract. New boundary conditions for classical integrable nonlinear lattices of the XXX type, such as the Heisenberg chain and the Toda lattice, are presented. These integrable extensions are formulated in terms of a generic XXX Heisenberg magnet interacting with two additional spins at each end of the chain. The construction uses the most general rank-1 ansatz for the 2×2 L-operator satisfying the reflection equation algebra with rational r-matrix. The associated quadratic algebra is shown to be that of dynamical symmetry for the A_1 and BC_2 Calogero–Moser problems. Other physical realizations of our quadratic algebra are also considered.

1. Introduction

Many of the Liouville integrable lattices, for instance the Toda lattice and the Heisenberg chain, remain integrable when imposing some boundary conditions other than the open or periodic ones. Usually it corresponds to switching from the A_{n-1} classical root system to the other ones (B_n, C_n, D_n, BC_n, etc) associated with an integrable lattice [1–3]. It was an idea of Sklyanin [1] (see also [4]) to describe all such possible boundary conditions in terms of the representations of a new algebra

$$R(u-v)T^{(1)}(u)R(u+v)T^{(2)}(v) = T^{(2)}(v)R(u+v)T^{(1)}(u)R(u-v) \qquad (1.1)$$

which was afterwards called *the reflection equation algebra* (see for instance [5]). We prefer to give it another name the *QISM II algebra* (see [2, 6, 7, 8]), where QISM stands for the quantum inverse scattering method and II symbolizes the difference between it and the *QISM I algebra*, which is simply the familiar algebra given by the quadratic relation [5, 9, 10, 11]

$$R(u-v)T^{(1)}(u)T^{(2)}(v) = T^{(2)}(v)T^{(1)}(u)R(u-v). \qquad (1.2)$$

There are actually too many names for these algebras, and none of them is becoming standard; that is why we insist on our own choices, the QISM I and II algebras. These names are short and also quite characteristic. We should also mention [12] where a different version of the reflection equation (1.1) was considered.

In terms of the integrable models the representations of the QISM I algebra (1.2) provide us with integrable lattices (through the co-multiplication) while the representations of the QISM II algebra (1.1) describe the possible boundary conditions for such lattices. In this paper

we use the new representation of the QISM II algebra found recently in [8] to obtain some new boundary terms for the known integrable lattices. These new models are generically formulated in terms of the XXX Heisenberg magnet interacting with two additional spins on each end. We also consider the degenerate cases given by the corresponding contraction procedure. We would like to stress that our results give for the moment the most general boundary terms for the Heisenberg magnet (expressed in terms of two additional spins on each end) and specialization will give already known boundary conditions [1, 2] coming from the scalar or rank-0 solutions of the QISM II algebra. Describing the situation in rather more technical terms, we can say that we are classifying all boundary conditions coming from the most general rank-1 representations of the QISM II algebra in the XXX case. All results in this paper are given for the Poisson algebra. The quantization is straightforward and will be published elsewhere.

2. Rank-1 representations of the QISM II algebra

In the classical limit the QISM II algebra (1.1) becomes the following Poisson algebra [1]:

$$\{T^{(1)}(u), T^{(2)}(v)\} = [r(u-v), T^{(1)}(u)T^{(2)}(v)] + T^{(1)}(u)r(u+v)T^{(2)}(v)$$
$$-T^{(2)}(v)r(u+v)T^{(1)}(u) \qquad (2.1)$$

where $T^{(1)}(u) \equiv T(u) \otimes I$, $T^{(2)}(v) \equiv I \otimes T(v)$. Here $\otimes$ denotes the usual tensor product and I is the 2×2 identity matrix. The $T(u)$ is the 2×2 monodromy matrix depending on the complex spectral parameter u. The 4×4 matrix $r(u)$ will, in this paper, be given by

$$r(u) = \frac{-1}{u} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (2.2)$$

This is the rational case, which corresponds to the XXX model [3, 9].

Let $T(u)$ be of the following form [7, 8]:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \qquad (2.3)$$

with

$$A(u) = \alpha u^2 + A_1 u + A_0 + \frac{\delta}{u} \qquad D(u) = -A(-u) \qquad (2.4)$$

$$B(u) = \beta u^2 + B_0 \qquad C(u) = \gamma u^2 + C_0 . \qquad (2.5)$$

Here α, β, γ, and δ are scalars, while A_1, A_0, B_0, and C_0 are generators of some algebra. The $T(u)$ (2.3)–(2.5) satisfies the following symmetry property:

$$T(-u) \sim T^{-1}(u) . \qquad (2.6)$$

Inserting this ansatz for $T(u)$ into the Poisson algebra (2.1) leads to the following quadratic Poisson algebra $\mathcal{A}$ for the generators A_1, A_0, B_0, and C_0:

$$\{A_0, A_1\} = \beta C_0 - \gamma B_0 \qquad \{B_0, A_0\} = 2A_1 B_0 - 2\beta\delta \qquad (2.7)$$

$$\{B_0, A_1\} = 2\alpha B_0 - 2\beta A_0 \qquad \{C_0, A_0\} = -2A_1 C_0 + 2\gamma\delta \qquad (2.8)$$

$$\{C_0, A_1\} = 2\gamma A_0 - 2\alpha C_0 \qquad \{C_0, B_0\} = 4A_1 A_0 - 4\alpha\delta . \qquad (2.9)$$

The determinant of the monodromy matrix $T(u)$ is the generating function for the centre of the QISM II algebra (2.1). For the ansatz (2.4)–(2.5)

$$\det T(u) = -(\alpha^2 + \beta\gamma)u^4 + Q_2 u^2 + Q_0 + \frac{\delta^2}{u^2} \tag{2.10}$$

with

$$Q_2 = A_1^2 - 2\alpha A_0 - \beta C_0 - \gamma B_0 \qquad Q_0 = 2\delta A_1 - A_0^2 - B_0 C_0. \tag{2.11}$$

Hence (2.11) gives two Casimir elements for the algebra (2.7)–(2.9). Notice here that we term the ansatz (2.4)–(2.5) the 'rank-1 representation', since we have got the Poisson algebra (2.7)–(2.9) with six generators out of it, which has two Casimir elements. So the rank of the algebra is equal to 1. If one takes higher order polynomials in the expressions for $A(u)$, $B(u)$, $C(u)$, $D(u)$, then one arrives at the Poisson algebras of a higher rank.

The QISM II algebra (2.1) with the r-matrix (2.2) admits the following scalar solution [1, 2]:

$$K(u) = \begin{pmatrix} a + d/u & b \\ c & -a + d/u \end{pmatrix} \tag{2.12}$$

where a, b, c, and d are complex constants. This is just a special case of (2.3)–(2.5) with $\alpha = \beta = \gamma = A_1 = 0$. Then A_0, B_0, and C_0 all commute. We may now combine the two solutions $K(u)$ and $T(u)$ of the QISM II algebra and define

$$t(u) = \operatorname{tr} K^t(-u)T(u). \tag{2.13}$$

Note that $Z : T(u) \mapsto T^t(-u)$ is an automorphism of the algebra (2.1) with the r-matrix (2.2). It is a property of the QISM II algebra [1] that

$$\{t(u), t(v)\} = 0. \tag{2.14}$$

Hence the $t(u)$ (2.13) is the generating function for the integrals of motion of an associated integrable system. For our special choices of $K(u)$ and $T(u)$ we get

$$t(u) = (2\alpha a + \beta b + \gamma c)u^2 + H - 2d\frac{\delta}{u^2} \tag{2.15}$$

where

$$H = 2aA_0 + bB_0 + cC_0 - 2dA_1. \tag{2.16}$$

The Hamiltonian (2.16) defines a completely integrable system, since it is effectively one dimensional. (Subtract two centre elements (2.11) from four generators and divide by two to see that the algebra (2.7)–(2.9) is of one degree of freedom.) In the appendix we show how we may eliminate the first term in the Hamiltonian (2.16) by using the automorphisms of the quadratic algebra $\mathcal{A}$. The result is

$$H = \tilde{b}B_0 + \tilde{c}C_0 - 2dA_1 \tag{2.17}$$

where the new coefficients $\tilde{b}$ and $\tilde{c}$ are related to a, b, and c through

$$a^2 + bc = \tilde{b}\tilde{c} \qquad 2\alpha a + \beta b + \gamma c = \beta\tilde{b} + \gamma\tilde{c}. \tag{2.18}$$

Any Hamiltonian that is a function of A_1, A_0, B_0, and C_0 will be completely integrable. However, for this special case, an arbitrary linear combination, we have formulated the problem using the QISM II algebra. It is then possible to apply the method of separation of variables, which will be shown in section 4. We now give a physical realization of the quadratic algebra (2.7)–(2.9).

3. The o(4) generalized Lagrange top

The algebra (2.7)–(2.9) may be embedded into the $\mathcal{U}(o(4))$ Lie algebra with the six generators J_i and x_i, $i = 1, 2, 3$, and the Poisson brackets

$$\{J_i, J_j\} = \epsilon_{ijk} J_k \qquad \{J_i, x_j\} = \epsilon_{ijk} x_k \qquad \{x_i, x_j\} = \epsilon_{ijk} J_k . \qquad (3.1)$$

The $\mathcal{U}(o(4))$ algebra has the two Casimir elements

$$C_1 = x^2 + J^2 \qquad C_2 = x \cdot J . \qquad (3.2)$$

The homomorphism between the $\mathcal{U}(o(4))$ algebra with relations (3.1) and the rank-1 QISM II algebra (2.7)–(2.9) is given by the following formulae [8]:

$$A_0 = x_1 J_2 - x_2 J_1 \qquad B_0 = -x_1^2 - x_2^2 - J_3^2 \qquad C_0 = J^2 \qquad A_1 = x_3 \qquad (3.3)$$

where the scalars α, β, and γ appearing in (2.4)–(2.5) are without loss of generality chosen to be

$$\alpha = 0 \qquad \beta = -1 \qquad \gamma = 1 \qquad\qquad\qquad (3.4)$$

with δ being equal to

$$\delta = C_2 J_3 . \qquad (3.5)$$

Using equations (2.10), (2.11) we may write the centre of the algebra in terms of the o(4) variables

$$\mathcal{Q}_0 = C_2^2 + C_1 J_3^2 \qquad \mathcal{Q}_2 = C_1 + J_3^2 . \qquad (3.6)$$

Notice that from this it follows that J_3 commutes with all of A_1, A_0, B_0, and C_0. It is a simple matter to verify equations (2.7)–(2.9) from the definitions (3.3) and (3.1). The integrable Hamiltonian (2.17) becomes

$$H = -\tilde{b} \mathcal{Q}_2 + (\tilde{b} + \tilde{c}) J^2 + \tilde{b} x_3^2 - 2 d x_3 \qquad (3.7)$$

with J_3 as the additional conserved quantity. For the particular values of α, β, and γ in (3.4) it can be shown that $\tilde{b}$ and $\tilde{c}$ are always real when a, b, and c are. The Hamiltonian (3.7) corresponds to an o(4) generalized Lagrange top [3].

   In the special case $\alpha^2 + \beta\gamma = 0$, the algebra $\mathcal{A}$ is degenerate. We then have the following homomorphism to $\mathcal{U}(e(3))$ [8]:

$$A_0 = x_1 J_2 - x_2 J_1 \qquad B_0 = -x_1^2 - x_2^2 \qquad C_0 = J^2 \qquad A_1 = x_3 \qquad (3.8)$$

where we have put without loss of generality that $\alpha = \beta = 0$ and $\gamma = 1$ with δ being as in (3.5). The algebra generators J_i and x_i satisfy here the Poisson brackets like in (3.1) where the last bracket is equal zero (according to the contraction procedure from o(4) to e(3)). The first Casimir element $C_1 = x^2$. The Hamiltonian (3.7) for this case corresponds to the Lagrange top in a nonlinear gravity field [3].

4. Separation of variables

We now return to the Hamiltonian (2.16) with the Poisson brackets (2.7)–(2.9) and show how this system may be integrated by the general method of separation of variables applicable to the QISM II algebra. This method was first applied by Sklyanin in [11, 13] to the representations of the QISM I algebra. See also [2, 6] where the separation of variables was applied for the first time to the QISM II algebra under circumstances close to the ones we have in the present paper.

To do this it is necessary first to apply a similarity transformation to the matrices $K(u)$ and $T(u)$ in order to make $K(u)$ triangular. We define the matrix V as

$$V = \begin{pmatrix} a+w & 0 \\ b & 1 \end{pmatrix} \tag{4.1}$$

where a and b are the parameters in $K(u)$ defined by (2.12) and $w^2 = a^2 + bc$. Then we introduce $\tilde{K}(u)$ and $\tilde{T}(u)$ as

$$\tilde{K}(u) = V^t K(u)(V^{-1})^t \qquad \tilde{T}(u) = V^{-1}T(u)V = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & -\tilde{A}(-u) \end{pmatrix}. \tag{4.2}$$

The matrices $\tilde{K}(u)$ and $\tilde{T}(u)$ are of the same form as $K(u)$ and $T(u)$, respectively. The $\tilde{K}(u)$ now looks like

$$\tilde{K}(u) = \begin{pmatrix} w+\dfrac{d}{u} & 0 \\ \dfrac{c}{a+w} & -w+\dfrac{d}{u} \end{pmatrix} = \begin{pmatrix} \tilde{a}+\dfrac{\tilde{d}}{u} & \tilde{b} \\ \tilde{c} & -\tilde{a}+\dfrac{\tilde{d}}{u} \end{pmatrix} \tag{4.3}$$

while we have the following relations between the entries of $T(u)$ and $\tilde{T}(u)$:

$$\tilde{\alpha} = \alpha + \frac{\beta b}{a+w} \qquad \tilde{\beta} = \frac{\beta}{a+w} \qquad \tilde{\gamma} = -2b\alpha - \frac{\beta b^2}{a+w} + (a+w)\gamma \tag{4.4}$$

and

$$A_0 = \tilde{A}_0 - b\tilde{B}_0 \qquad B_0 = (a+w)\tilde{B}_0 \qquad C_0 = \frac{2b\tilde{A}_0 - b^2\tilde{B}_0 + \tilde{C}_0}{a+w} \tag{4.5}$$

where A_1 and δ are not changed. Since $\det \tilde{T}(u) = \det T(u)$ we have $\tilde{\mathcal{Q}}_2 = \mathcal{Q}_2$ and $\tilde{\mathcal{Q}}_0 = \mathcal{Q}_0$. Furthermore, $\tilde{K}(u)$ and $\tilde{T}(u)$ also satisfy the QISM II algebra, while $t(u)$ defined in (2.13) is unaltered. Hence, these matrices generate the same integrable system as before. It is because of the triangular form of $\tilde{K}(u)$ that we get

$$t(u) = \tilde{a}(\tilde{A}(u) + \tilde{A}(-u)) - \frac{\tilde{d}}{u}(\tilde{A}(u) - \tilde{A}(-u)) + \tilde{c}\tilde{C}(u). \tag{4.6}$$

Now let u_1 be a zero of $\tilde{C}(u)$, i.e.

$$\tilde{C}(u_1) = 0 \tag{4.7}$$

and define

$$\lambda_1^+ = -\tilde{A}(-u_1) \qquad \lambda_1^- = \tilde{A}(u_1). \tag{4.8}$$

We may easily evaluate

$$\lambda_1^+\lambda_1^- = -\tilde{A}(u_1)\tilde{A}(-u_1) = \det \tilde{T}(u_1) = \Delta(u_1) \tag{4.9}$$

where $\Delta(u) = \det T(u)$ is given by (2.10), and from (4.6) we get

$$t(u_1) = w(\lambda_1^- - \lambda_1^+) - \frac{d}{u_1}(\lambda_1^- + \lambda_1^+). \tag{4.10}$$

This is simply the separation equation for the separation variables $u_1, \lambda_1^\pm$. Because $\tilde{T}(u)$ satisfies the QISM II algebra we have the important Poisson brackets between the new variables u_1 and $\lambda_1^\pm$ (see [2, 6])

$$\{u_1, \lambda_1^\pm\} = \pm\lambda_1^\pm \qquad \{\lambda_1^+, \lambda_1^-\} = -\Delta'(u_1). \tag{4.11}$$

From (4.9) and (4.11) it follows that we may put

$$\lambda_1^{\pm} = \sqrt{\Delta(u_1)}\, e^{\pm p_1} \qquad \{u_1, p_1\} = 1 . \tag{4.12}$$

Inserting (4.12) into (4.10) and using (2.15) gives

$$H = -2\sqrt{\Delta(u_1)} \left(w \sinh p_1 + \frac{d}{u_1} \cosh p_1 \right) - (2\alpha a + \beta b + \gamma c)\, u_1^2 + 2d \frac{\delta}{u_1^2} . \tag{4.13}$$

We have thus formulated the problem as an one-dimensional system with the Hamiltonian (4.13) in terms of the canonical coordinates u_1 and p_1. The equation of motion looks like

$$\dot{u}_1 = \{u_1, H\} \tag{4.14}$$

from which we have

$$(\dot{u}_1)^2 = t(u_1)^2 - 4\Delta(u_1) \left(\frac{d^2}{u_1^2} - w^2 \right) . \tag{4.15}$$

After the substitution $v_1 = u_1^2$ the problem can be integrated in terms of elliptic functions. The transformation from (A_1, A_0, B_0, C_0) to (u_1, p_1) is given by equations (2.16), (4.7), and (4.13). To get back we first write $\tilde{A}(u)$ and $\tilde{C}(u)$ in terms of u_1 and p_1 through the Lagrange interpolation using the data

$$\tilde{C}(\pm u_1) = 0 \qquad \tilde{A}(\pm u_1) = \pm \lambda_1^{\mp} \tag{4.16}$$

and the leading parameters $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, and $\tilde{\delta}$. This gives

$$\tilde{C}(u) = \tilde{\gamma}(u^2 - u_1^2) \tag{4.17}$$

$$\tilde{A}(u) = \tilde{\alpha}(u^2 - u_1^2) + \frac{\tilde{\delta}}{u} \frac{u_1^2 - u^2}{u_1^2} + \frac{u - u_1}{2u_1}\lambda_1^+ + \frac{u + u_1}{2u_1}\lambda_1^- . \tag{4.18}$$

Equating powers of u and using (4.12) then gives

$$\tilde{C}_0 = -\tilde{\gamma}u_1^2 \qquad \tilde{A}_0 = -\tilde{\alpha}u_1^2 - \sqrt{\Delta(u_1)} \sinh p_1 \qquad \tilde{A}_1 = -\frac{\tilde{\delta}}{u_1^2} + \frac{\sqrt{\Delta(u_1)}}{u_1} \cosh p_1 \tag{4.19}$$

To get $\tilde{B}_0$ we insert $\tilde{B}(u) = \tilde{\beta}u^2 + \tilde{B}_0$ into $-\tilde{A}(u)\tilde{A}(-u) - \tilde{B}(u)\tilde{C}(u) = \Delta(u)$ and equate powers of u. This gives

$$\tilde{B}_0 = \frac{1}{\tilde{\gamma}} \left(2\frac{\tilde{\delta}^2}{u_1^4} + \frac{\tilde{Q}_0}{u_1^2} + \tilde{\alpha}^2 u_1^2 + \frac{\Delta(u_1)}{u_1^2} \sinh^2 p_1 + 2\sqrt{\Delta(u_1)} \left(\tilde{\alpha} \sinh p_1 - \frac{\tilde{\delta}}{u_1^3} \cosh p_1 \right) \right) . \tag{4.20}$$

Alternatively, $\tilde{B}_0$ may be restored from the Poisson bracket

$$\{\tilde{A}_0, \tilde{A}_1\} = \tilde{\beta}\tilde{C}_0 - \tilde{\gamma}\tilde{B}_0. \tag{4.21}$$

We may finally get A_1, A_0, B_0, and C_0 in terms of u_1, p_1, Q_0, and Q_2 by using (4.5).

5. Two interacting o(4) tops

We now generalize the results of sections 2 and 3 to the case of not just one, but two o(4) Lagrange tops interacting with each other. This is done by replacing in (2.13) $K(u)$ with $\tilde{T}(u)$, where $\tilde{T}(u)$ is of the same form as $T(u)$, but with a different choice of $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$, $\tilde{\delta}$, $\tilde{A}_1$, $\tilde{A}_0$, $\tilde{B}_0$, and $\tilde{C}_0$. Because the QISM II algebra (2.1) is closed under the action of any similarity transformation, we may write $t(u)$ as

$$t(u) = \operatorname{tr} \tilde{T}^t(-u) V T(u) V^{-1} \tag{5.1}$$

where V is any matrix with determinant 1, while $T(u)$ is given by (2.3)–(2.5) and (3.3)–(3.5) and similarly for $\tilde{T}(u)$. Putting

$$V = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5.2}$$

with $ad - bc = 1$ leads to the following integrable system:

$$t(u) = (a^2 + b^2 + c^2 + d^2)u^4 - Hu^2 - G - \frac{2\delta\tilde{\delta}}{u^2} \tag{5.3}$$

where

$$
\begin{aligned}
H = &-2(ab + cd)A_0 + (a^2 + c^2)B_0 - (b^2 + d^2)C_0 \\
&-2(ac + bd)\tilde{A}_0 + (a^2 + b^2)\tilde{B}_0 - (c^2 + d^2)\tilde{C}_0 + 2A_1\tilde{A}_1
\end{aligned} \tag{5.4}
$$

$$
\begin{aligned}
G = &-2(1 + 2bc)A_0\tilde{A}_0 - a^2 B_0\tilde{B}_0 - d^2 C_0\tilde{C}_0 + 2a(c B_0\tilde{A}_0 + b A_0\tilde{B}_0) \\
&+b^2 C_0\tilde{B}_0 + c^2 B_0\tilde{C}_0 - 2d(b C_0\tilde{A}_0 + c A_0\tilde{C}_0) + 2(\delta\tilde{A}_1 + \tilde{\delta}A_1).
\end{aligned} \tag{5.5}
$$

If we regard the H (5.4) as the Hamiltonian then we see that the interaction between the two systems is through the term $A_1\tilde{A}_1$. We may use the automorphisms of the quadratic algebra $\mathcal{A}$ to eliminate the terms in H with A_0 and $\tilde{A}_0$. Using equations (2.18) we get the following Hamiltonian:

$$H = \lambda(C_0 - \tilde{B}_0) + \lambda^{-1}(\tilde{C}_0 - B_0) + 2A_1\tilde{A}_1 \tag{5.6}$$

where λ is the single non-trivial parameter in the Hamiltonian satisfying $-(\lambda + \lambda^{-1}) = a^2 + b^2 + c^2 + d^2$. The second conserved quantity is

$$G = 2(A_0\tilde{A}_0 + \delta\tilde{A}_1 + \tilde{\delta}A_1) - \lambda C_0\tilde{B}_0 - \lambda^{-1}B_0\tilde{C}_0. \tag{5.7}$$

Applying the homomorphism (3.3) gives the final result:

$$H = J^2 + \tilde{x}^2 + (x_3 \cosh\theta + \tilde{x}_3 \sinh\theta)^2. \tag{5.8}$$

Here $\lambda = \tanh\theta$, while constant additive terms and multiplicative factors have been neglected. This is a new integrable system which we call *two interacting o(4) Lagrange tops*. We remark that the number of degrees of freedom is 4 and there are two simple integrals, namely: J_3 and $\tilde{J}_3$. The fourth integral G (5.7) is of degree 4 in the J_i and x_i. No separation of variables is known for this system or in general for any integrable system given by (5.1) in the case when both $T(u)$ and $\tilde{T}(u)$ are non-scalar.

The Hamiltonian (5.8) contains interesting subcases. We may easily change the real form speaking about two interacting systems on the algebras, for instance, o(2, 2) ⊕ o(2, 2), o(3, 1) ⊕ o(4) and so on. The further possibility is the contraction giving the cases like: e(3) ⊕ o(4), e(3) ⊕ o(3, 1), e(3) ⊕ e(3) etc. Finally, the two o(4) algebras have two o(3)

subalgebras by putting $(x_1, J_2, x_3) = (s_1, s_2, s_3)$ and similarly $(\tilde{x}_1, \tilde{J}_2, \tilde{x}_3) = (t_2, -t_1, t_3)$.
Then

$$H = \lambda(s_2^2 + t_2^2) + \lambda^{-1}(s_1^2 + t_1^2) + 2s_3 t_3 \tag{5.9}$$

$$G = \lambda^{-1}(s_1 t_1 - \lambda s_2 t_2)^2. \tag{5.10}$$

This is a special (but non-trivial, one-parameter) case of the general (two parameter) o(4)
Manakov top [2, 3].

6. Further realizations

In this section we consider other homomorphisms of the specialized quadratic algebra $\mathcal{A}$
determined by the condition $\alpha^2 + \beta\gamma = 0$. Without loss of generality we may put $\alpha = \beta = 0$
and $\gamma = 1$. Applying the techniques of the section 5 leads to the following system:

$$H = C_0 + \tilde{C}_0 + 2\lambda A_1 \tilde{A}_1 \tag{6.1}$$

$$G = C_0 \tilde{C}_0 + 2\lambda(\delta \tilde{A}_1 + \tilde{\delta} A_1 - A_0 \tilde{A}_0) + \lambda^2 B_0 \tilde{B}_0 \tag{6.2}$$

with the single parameter λ. We have the following homomorphism of the specialized
algebra $\mathcal{A}$ into $\mathcal{U}(sl(2) \oplus sl(2))$:

$$A_0 = 4(s_3 t_- - t_3 s_-) \qquad B_0 = -16t_- s_- \tag{6.3}$$

$$C_0 = (s_+ + t_+)(s_- + t_-) - (s_3 + t_3)^2 \qquad A_1 = 2(t_- - s_-) \tag{6.4}$$

$$\delta = 2(s_- + t_-)(C_s - C_t). \tag{6.5}$$

Here C_s and C_t are the Casimir elements of the two $sl(2)$ algebras

$$C_s = s_3^2 - s_+ s_- \qquad C_t = t_3^2 - t_+ t_- \tag{6.6}$$

and the Poisson brackets for the generators are given by

$$\{s_-, s_3\} = s_- \qquad \{s_-, s_+\} = 2s_3 \qquad \{s_3, s_+\} = s_+ \tag{6.7}$$

$$\{t_-, t_3\} = t_- \qquad \{t_-, t_+\} = 2t_3, \qquad \{t_3, t_+\} = t_+. \tag{6.8}$$

We may realize the pair of $sl(2)$ algebras in terms of the canonical variables (x, y, p_x, p_y)
via the following homomorphism ($\{x, p_x\} = 1$ etc):

$$s_3 = -\frac{xp_x}{2} \qquad s_- = \frac{x^2}{2} \qquad s_+ = \frac{p_x^2}{2} - \frac{2l}{x^2} \tag{6.9}$$

$$t_3 = -\frac{yp_y}{2} \qquad t_- = \frac{y^2}{2} \qquad t_+ = \frac{p_y^2}{2} - \frac{2m}{y^2} \tag{6.10}$$

where l and m are the values of C_s and C_t, respectively. Inserting these relations in (6.1)
gives the following Hamiltonian:

$$H = \tfrac{1}{4}(xp_y - yp_x)^2 + \tfrac{1}{4}(\tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x)^2 + 2\lambda(x^2 - y^2)(\tilde{x}^2 - \tilde{y}^2)$$

$$-(x^2 + y^2)\left(\frac{l}{x^2} + \frac{m}{y^2}\right) - (\tilde{x}^2 + \tilde{y}^2)\left(\frac{\tilde{l}}{\tilde{x}^2} + \frac{\tilde{m}}{\tilde{y}^2}\right). \tag{6.11}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $\tilde{x} = \tilde{r}\cos\tilde{\theta}$, $\tilde{y} = \tilde{r}\sin\tilde{\theta}$ this system becomes

$$H = \tfrac{1}{4}\dot{\theta}^2 + \tfrac{1}{4}\dot{\tilde{\theta}}^2 + 2\lambda r^2\tilde{r}^2\cos 2\theta\cos 2\tilde{\theta} - \frac{l}{\cos^2\theta} - \frac{m}{\sin^2\theta} - \frac{\tilde{l}}{\cos^2\tilde{\theta}} - \frac{\tilde{m}}{\sin^2\tilde{\theta}}$$

(6.12)

$$G = \tfrac{1}{16}\left(\dot{\theta}\dot{\tilde{\theta}} - 2\lambda r^2\tilde{r}^2\cos 2(\theta - \tilde{\theta}) + 2\lambda r^2\tilde{r}^2\cos 2(\theta + \tilde{\theta})\right)^2$$

$$+\left(\tfrac{1}{4}\dot{\theta}^2 - \frac{l}{\cos^2\theta} - \frac{m}{\sin^2\theta}\right)\left(\tfrac{1}{4}\dot{\tilde{\theta}}^2 - \frac{\tilde{l}}{\cos^2\tilde{\theta}} - \frac{\tilde{m}}{\sin^2\tilde{\theta}}\right) - \left(\frac{\dot{\theta}\dot{\tilde{\theta}}}{4}\right)^2$$

$$+2\lambda r^2\tilde{r}^2\left((m - l)\cos 2\tilde{\theta} + (\tilde{m} - \tilde{l})\cos 2\theta\right).$$

(6.13)

Regarding H, equation (6.12) as the Hamiltonian, we can see that we have a system of two particles situated in some singular field and interacting with each other through the product of cosines. This can be interpreted as two interacting pendulas. This is a new integrable system. The r^2 and $\tilde{r}^2$ are constants, being the level values of two simple integrals: $s_- + t_-$ and $\tilde{s}_- + \tilde{t}_-$. The fourth integral G, equation (6.13), is of degree 4 in terms of $\dot{\theta}$ and $\dot{\tilde{\theta}}$. We would like to call this system *the D_2 Toda lattice with singular terms*. Without the singular terms the system is easily integrated by introducing $\theta_\pm = \theta \pm \tilde{\theta}$. However, no integration of the general system is yet known. In what follows we will give two more physical examples connected with some realizations of the specialized algebra $\mathcal{A}$: namely, the so-called A_1 and BC_2 rational Calogero–Moser systems [14].

The A_1 rational Calogero–Moser system is given by the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \frac{1}{(x_1 - x_2)^2} \qquad \{p_i, x_j\} = \delta_{ij}$$

(6.14)

with the additional conserved quantity $P = p_1 + p_2$. Introduce three more variables

$$R = x_1 + x_2 \qquad H_0 = \tfrac{1}{2}(p_1 x_1 + p_2 x_2) \qquad H_+ = \tfrac{1}{2}(x_1^2 + x_2^2).$$

(6.15)

We now have the following realization of the specialized algebra $\mathcal{A}$:

$$A_0 = 4(PH_0 - RH) \qquad B_0 = 4(P^2 - 4H) \qquad C_0 = 4(HH_+ - H_0^2)$$

(6.16)

$$A_1 = 2P \qquad \delta = 0$$

(6.17)

again with $\alpha = \beta = 0$ and $\gamma = 1$. Since $\mathcal{Q}_2 = 16H$, H commutes with all of A_0, B_0, C_0, and A_1. This observation corresponds to the fact that the Hamiltonian (6.14) is superintegrable [15].

Inserting this realization in the Hamiltonian (6.1) for two 'interacting A_1 Calogero–Moser systems' leads to the following integrable system:

$$H = (x_1 p_2 - x_2 p_1)^2 + 2\frac{x_1^2 + x_2^2}{(x_1 - x_2)^2} + (\tilde{x}_1\tilde{p}_2 - \tilde{p}_2\tilde{x}_1)^2$$

$$+2\frac{\tilde{x}_1^2 + \tilde{x}_2^2}{(\tilde{x}_1 - \tilde{x}_2)^2} + \lambda(p_1 + p_2)(\tilde{p}_1 + \tilde{p}_2).$$

(6.18)

By the construction of this system, we know that its motion on the levels of two Hamiltonians for two A_1 Calogero–Moser systems is equivalent to that of the system (6.12) (because both systems give two different realizations of the *same* specialized algebra $\mathcal{A}$).

Consider finally the following one-parameter case of the BC_2 rational Calogero–Moser system given by the Hamiltonian:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 + x_2)^2} + \frac{\kappa^2}{2}\left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right) \tag{6.19}$$

where κ is a constant. This system may be embedded into the specialized algebra $\mathcal{A}$ in the following manner:

$$C_0 = \tfrac{1}{4}(HH_+ - H_0^2) \qquad B_0 = 4(N - 4H^2)(N - 2H^2) \tag{6.20}$$

$$A_0 = \{N, C_0\} \qquad A_1 = -2(N - 3H^2) \qquad \delta = (\kappa^2 - 1)H^2 \tag{6.21}$$

where H_0 and H_+ are given by (6.15) and N is the so-called second conserved quantity [14] ($\{H, N\} = 0$)

$$N = p_1^4 + p_2^4 + 2\kappa^2\left(\frac{p_1^2}{x_1^2} + \frac{p_2^2}{x_2^2}\right) + 8(p_1^2 + p_2^2)\frac{x_1^2 + x_2^2}{(x_1^2 - x_2^2)^2}$$

$$+16\frac{x_1 x_2 p_1 p_2}{(x_1^2 - x_2^2)^2} + \frac{(x_1^4 + x_2^4)(\kappa^2(x_1^2 - x_2^2)^2 + 4x_1^2 x_2^2)^2}{(x_1^2 - x_2^2)^4 x_1^4 x_2^4}. \tag{6.22}$$

Because the Hamiltonian H is sitting in δ it commutes with all the generators and hence the system (6.19) is superintegrable too. In this case the Hamiltonian (6.1) for 'two interacting BC_2 Calogero–Moser systems' has the following form:

$$H = (x_1 p_2 - x_2 p_1)^2 + (\tilde{x}_1 \tilde{p}_2 - \tilde{x}_2 \tilde{p}_1)^2 + 2(x_1^2 + x_2^2)U(x_1, x_2, \kappa)$$

$$+2(\tilde{x}_1^2 + \tilde{x}_2^2)U(\tilde{x}_1, \tilde{x}_2, \tilde{\kappa}) + \lambda(N - 3H^2)(\tilde{N} - 3\tilde{H}^2) \tag{6.23}$$

where

$$U(x_1, x_2, \kappa) = \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 + x_2)^2} + \frac{\kappa^2}{2}\left(\frac{1}{x_1^2} + \frac{1}{x_2^2}\right). \tag{6.24}$$

It is quite interesting to remark that the motion given by the Hamiltonian (6.23) on the fixed levels of two integrals of motion (6.19) (one with tilde and one without tilde) is the same as the motion given by the Hamiltonian of the D_2 Toda lattice (6.12). The last two results regarding the Calogero–Moser systems were obtained by one of us in [16].

7. Two e(3) tops interacting with the A_n Toda lattice

The next generalization we present is that of inserting the Toda lattice between the two e(3) Lagrange tops (see end of section 3). This is done by using the monodromy matrix for the Toda lattice

$$L(u) = L_2(u) \cdots L_{N-1}(u) \tag{7.1}$$

where

$$L_i(u) = \begin{pmatrix} 0 & \exp(q_i) \\ \exp(-q_i) & u - p_i \end{pmatrix} \qquad \{q_i, p_j\} = \delta_{ij}. \tag{7.2}$$

The matrix $L(u)$ (7.1) satisfies the Poisson limit of the QISM I algebra [9, 13]

$$\{L^{(1)}(u), L^{(2)}(v)\} = [r(u - v), L^{(1)}(u)L^{(2)}(v)] \tag{7.3}$$

with the same r-matrix (2.2) as in (2.1). Now let $T_1(u)$ and $T_N(u)$ be the representations described in (3.8). It is then true that the matrix

$$T(u) = L(u)T_N(u)L^{-1}(-u) \tag{7.4}$$

also satisfies the QISM II algebra (2.1) [1]. Therefore the trace

$$t(u) = \operatorname{tr} T_1^t(-u)L(u)T_N(u)L^{-1}(-u) \tag{7.5}$$

is a generating function for the integrals of motion for the A_{n-2} Toda lattice interacting with an e(3) Lagrange top at each end. The Hamiltonian for this system is as follows:

$$H = \frac{1}{2}\sum_{i=2}^{N-1} p_i^2 + J_1^2 + J_N^2 - \sum_{i=2}^{N-2} \exp(q_{i+1} - q_i) - e^{q_2}x_{1,3} + e^{q_{N-1}}x_{N,3}. \tag{7.6}$$

The first three terms describe the kinetic energy of the system, which is a chain of particles plus two tops at the ends of the chain. The last three terms are the potentials, the first one being in the form of Toda-like interaction between the neighbours in the chain while the two last terms describe an interaction of the tops with the chain. It is convenient to choose the following representation of the specialized quadratic algebra $\mathcal{A}$. Let $\alpha = \beta = 0$ and $\gamma = -1$. Then (cf also [2, 6])

$$A_1 = \cosh q \qquad A_0 = p \sinh q \qquad B_0 = -\sinh^2 q \tag{7.7}$$

$$C_0 = p^2 + \frac{2c_1}{\sinh^2 \frac{q}{2}} + \frac{2c_2}{\cosh^2 \frac{q}{2}} \qquad \{p, q\} = 1 \tag{7.8}$$

where c_1 and c_2 are arbitrary constants. The Hamiltonian (7.6) then becomes

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 - \sum_{i=2}^{N-2} \exp(q_{i+1} - q_i) - e^{q_2}\cosh q_1 + e^{q_{N-1}}\cosh q_N \tag{7.9}$$

$$+\frac{c_1}{\sinh^2 \frac{q_1}{2}} + \frac{c_2}{\cosh^2 \frac{q_1}{2}} + \frac{c_3}{\sinh^2 \frac{q_N}{2}} + \frac{c_4}{\cosh^2 \frac{q_N}{2}}. \tag{7.10}$$

This Hamiltonian was proved to be integrable for the first time in [17]. It corresponds to the most general Toda lattice of the type D_n with four additional singular terms. Now we have given another proof for its integrability and also shown how it appears naturally from the combination of the standard A_{n-2} Toda lattice with two tops each interacting with the corresponding edge particle of the A_{n-2} Toda lattice.

8. Two o(4) tops interacting with the Heisenberg magnet

Let us consider the XXX Heisenberg magnet. It is given by the following construction. First, we introduce a chain of simple L-operators [9] $(k = 1, \ldots, N)$

$$L_k(u) = 1 + \frac{i}{u}\begin{pmatrix} s_k^3 & s_k^+ \\ s_k^- & -s_k^3 \end{pmatrix} \tag{8.1}$$

each of which satisfies the QISM I r-matrix algebra $(T(u) = L_k(u))$ with the r-matrix (2.2)

$$\{T^{(1)}(u), T^{(2)}(v)\} = [r(u - v), T^{(1)}(u)T^{(2)}(v)] \tag{8.2}$$

where the s-variables have the following Poisson brackets of the $sl(2)$ Lie algebra for any k:

$$\{s^3, s^\pm\} = \pm is^\pm \qquad \{s^+, s^-\} = 2is^3. \tag{8.3}$$

We will also use the real variables s_k^1 and s_k^2 defined as $s_k^{\pm} = s_k^1 \mp \mathrm{i} s_k^2$. The L-operators (8.1) have the following properties when conjugate and change the sign of the spectral parameter u:

$$L_k(-u) = \bar{L}_k^t(u) \qquad \bar{L}_k(u) = \sigma_2 L_k(u) \sigma_2 \qquad u \in \mathbb{R}. \tag{8.4}$$

The L-operators (8.1) can be also represented in the form of the scalar product between two vectors:

$$L_k(u) = 1 + \frac{\mathrm{i}}{u}(s_k, \sigma) \qquad s_k = (s_k^1, s_k^2, s_k^3) \tag{8.5}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ are the standard Pauli matrices.

The determinants of the L-operators are expressed through the Casimir elements of the $sl(2)$ algebras:

$$\det L_k(u) = 1 + \frac{c_k^2}{u^2} \qquad c_k^2 = (s_k^1)^2 + (s_k^2)^2 + (s_k^3)^2 \tag{8.6}$$

and have $\pm \mathrm{i} c_k$ as degeneration points. The monodromy matrix $T(u) = L_N(u) \cdots L_1(u)$ satisfies the algebra (8.2) too, while its trace

$$t(u) = \operatorname{tr} T(u) \qquad \{t(u), t(v)\} = 0$$

provides us with the complete set of the integrals of motion in involution for the XXX Heisenberg magnet. There is a well known rule [9] for writing down the local Hamiltonian for the model. Suppose that we have the homogeneous chain, i.e. all c_k are equal, $c_k = c$. Then one should calculate the logarithm of the generating function $t(u)$ at the common degenerate point of all the L-operators, i.e. $u = \mathrm{i} c$ in our case. So, the local Hamiltonian for the XXX Heisenberg chain looks like:

$$H_{\mathrm{loc}} = \log |t(\mathrm{i} c)|^2 = \sum_{k=1}^{N} \log \left[2 + \frac{2}{c^2}(s_k, s_{k+1}) \right]. \tag{8.7}$$

Let us now proceed further to introducing the boundary conditions for the chain with the Hamiltonian (8.7). First, we pick up a special representation of the QISM II algebra which was introduced in sections 2 and 3. Suppose we have the Lie algebra $so(3) \oplus so(2, 1)$ given by the following Poisson brackets for its six generators s_k and t_k:

$$\{s_i, s_j\} = \varepsilon_{ijk} s_k \tag{8.8}$$

$$\{t_1, t_2\} = -t_3 \qquad \{t_2, t_3\} = t_1 \qquad \{t_3, t_1\} = t_2. \tag{8.9}$$

The Casimir elements are

$$C_s = (s_1)^2 + (s_2)^2 + (s_3)^2 = s^2 \qquad C_t = -(t_1)^2 - (t_2)^2 + (t_3)^2 = t^2. \tag{8.10}$$

The following matrix:

$$T(u) = \begin{pmatrix} u(s_3 - t_3) + 2\mathrm{i}(s_1 t_2 - s_2 t_1) & \mathrm{i}(u^2 + s^2 + t^2 + 2 s_3 t_3 \\ \quad + \frac{(s_3 + t_3)(s^2 - t^2)}{u} & \quad - 2\mathrm{i}(s_1 t_1 + s_2 t_2)) \\ \mathrm{i}(u^2 + s^2 + t^2 + 2 s_3 t_3 & u(s_3 - t_3) - 2\mathrm{i}(s_1 t_2 - s_2 t_1) \\ \quad + 2\mathrm{i}(s_1 t_1 + s_2 t_2)) & \quad + \frac{(s_3 + t_3)(s^2 - t^2)}{u} \end{pmatrix} \tag{8.11}$$

gives a representation of the QISM II algebra (2.1).

Let us take the same representation with the tilde-variables $\tilde{s}_k$ and $\tilde{t}_k$ which are in direct sum to the spins s and t:

$$\{\tilde{s}_i, \tilde{s}_j\} = \varepsilon_{ijk}\tilde{s}_k \tag{8.12}$$

$$\{\tilde{t}_1, \tilde{t}_2\} = -\tilde{t}_3 \qquad \{\tilde{t}_2, \tilde{t}_3\} = \tilde{t}_1 \qquad \{\tilde{t}_3, \tilde{t}_1\} = \tilde{t}_2. \tag{8.13}$$

The Casimir elements are

$$\tilde{C}_s = (\tilde{s}_1)^2 + (\tilde{s}_2)^2 + (\tilde{s}_3)^2 = \tilde{s}^2 \qquad \tilde{C}_t = -(\tilde{t}_1)^2 - (\tilde{t}_2)^2 + (\tilde{t}_3)^2 = \tilde{t}^2. \tag{8.14}$$

The following matrix $\tilde{T}(u)$:

$$\tilde{T}(u) = \begin{pmatrix} u(\tilde{s}_3 - \tilde{t}_3) + 2i(\tilde{s}_1\tilde{t}_2 - \tilde{s}_2\tilde{t}_1) & i(u^2 + \tilde{s}^2 + \tilde{t}^2 + 2\tilde{s}_3\tilde{t}_3 \\ + \frac{(\tilde{s}_3+\tilde{t}_3)(\tilde{s}^2-\tilde{t}^2)}{u} & -2i(\tilde{s}_1\tilde{t}_1 + \tilde{s}_2\tilde{t}_2)) \\ i(u^2 + \tilde{s}^2 + \tilde{t}^2 + 2\tilde{s}_3\tilde{t}_3 & u(\tilde{s}_3 - \tilde{t}_3) - 2i(\tilde{s}_1\tilde{t}_2 - \tilde{s}_2\tilde{t}_1) \\ +2i(\tilde{s}_1\tilde{t}_1 + \tilde{s}_2\tilde{t}_2)) & + \frac{(\tilde{s}_3+\tilde{t}_3)(\tilde{s}^2-\tilde{t}^2)}{u} \end{pmatrix} \tag{8.15}$$

gives a representation of the QISM II algebra (2.1).

The generating function for the integrals of motion for the XXX Heisenberg chain with boundary terms looks like

$$t(u) = \mathrm{tr}\, \tilde{T}(u) L_N(u) \cdots L_1(u) T(u) L_1(u) \cdots L_N(u). \tag{8.16}$$

The local Hamiltonian for the system has the following form:

$$H_{\mathrm{loc}} = \log t(\mathrm{i}c) = \sum_{k=1}^{N-1} \log\left[2 + \frac{2}{c^2}(s_k, s_{k+1})\right]$$

$$+ \log[c^2(s_3 - t_3) - (s_3 + t_3)(s^2 - t^2) + s_1^1(-c^2 + s^2 + t^2 + 2s_3t_3)$$

$$+ 2s_1^2(s_1t_1 + s_2t_2) + 2s_1^3(s_1t_2 - s_2t_1)]$$

$$+ \log[c^2(\tilde{s}_3 - \tilde{t}_3) - (\tilde{s}_3 + \tilde{t}_3)(\tilde{s}^2 - \tilde{t}^2) + s_N^1(-c^2 + \tilde{s}^2 + \tilde{t}^2 + 2\tilde{s}_3\tilde{t}_3)$$

$$+ 2s_N^2(\tilde{s}_1\tilde{t}_1 + \tilde{s}_2\tilde{t}_2) + 2s_N^3(\tilde{s}_1\tilde{t}_2 - \tilde{s}_2\tilde{t}_1)] + \log\left(\frac{-4}{c^2}\right). \tag{8.17}$$

This integrable Hamiltonian describes the XXX Heisenberg chain interacting with two spins s and t at the one end of the chain and with two spins $\tilde{s}$ and $\tilde{t}$ at the other one. Note that in order to get the Hermitian Hamiltonians we had to choose the non-compact $so(2, 1)$ form of the spins t and $\tilde{t}$. The boundary terms in the Hamiltonian (8.17) generalize the ones in [18], which described the influence of the external magnetic field and, in quantum case, looked like (formula (53) in [18])

$$H = \sum_{n=1}^{N-1}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z) + b_-\sigma_1^z - b_+\sigma_N^z + c_-\sigma_1^- - c_+\sigma_N^-$$

$$+ d_-\sigma_1^+ - d_+\sigma_N^+ \tag{8.18}$$

where $b_\pm$, $c_\pm$, and $d_\pm$ are constants. The Hamiltonian similar to (8.18), with six boundary terms, was proved recently to be integrable also for the generalization up to the XYZ case [19].

9. Discussion

It is well known (after the original work of Sklyanin [1]) that every new representation of the reflection equation algebra, or the QISM II algebra in our terms, provides one with some new integrable system which can be interpreted as a system described by the QISM I algebra with some boundary terms added.

In this paper we constructed several new integrable systems which appear after imposing boundary conditions on the known integrable lattices. We obtained the following results: (i) two interacting o(4) tops; (ii) a new interpretation of the most general Toda lattice of the D_N type (namely, '$D_N = A_N + 2$ tops'); (iii) the quadratic algebra $\mathcal{A}$ as a dynamical algebra of hidden symmetries for the A_1 and BC_2 Calogero–Moser problems; (iv) the explicit form of the 'local' Hamiltonian (8.17) for the system which describes an interaction of the XXX Heisenberg chain with two o(4, $\mathbb{C}$) tops.

We believe that it is possible to generalize our results in some ways: first, the quantization which seems quite straightforward, and second, the q-deformation. There is also an open problem of whether to integrate new systems or to separate variables for them.

It is also very interesting to understand how many spins can be added to the Heisenberg chain as some sort of boundary conditons, while still preserving the integrability property. In the present paper we showed how to add two more spins at each end. Our conjecture is that it is possible to extend this up to *three* spins at each end of the standard spin chain. This conjecture came from the study of the hypergeometric orthogonal polynomials [8].

Appendix. Automorphisms of the quadratic algebra $\mathcal{A}$

In this appendix we discuss the automorphisms of the quadratic algebra $\mathcal{A}$ given by the relations (2.7)–(2.9). It is a property of the QISM II algebra (2.1) that any similarity transformation of a representation $T(u)$ is an automorphism of the QISM II algebra, i.e.

$$\tilde{T}(u) = V^{-1} T(u) V \tag{A.1}$$

for any non-degenerate matrix V satisfies the QISM II algebra too. Rewriting the ansatz (2.3)–(2.5) as

$$T(u) = u^2 \Omega + A_1 u I + X + \frac{\delta}{u} I \tag{A.2}$$

where I is the 2×2 identity matrix, and

$$\Omega = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \qquad X = \begin{pmatrix} A_0 & B_0 \\ C_0 & -A_0 \end{pmatrix} \tag{A.3}$$

it becomes clear that A_1 and δ do not change under the transformation (A.1). We wish to find a matrix V that leaves α, β, and γ unaltered. This will be so if V commutes with Ω. We consider therefore the following choice:

$$V(\theta) = \exp\left(\frac{\theta}{\Delta} \Omega\right) \tag{A.4}$$

where $\Delta^2 = \det \Omega = -(\alpha^2 + \beta\gamma)$. The only part of $T(u)$ that is changed is X, according to the following formula:

$$X(\theta) = V(-\theta) X V(\theta). \tag{A.5}$$

Relation (A.5) defines an automorphism of the quadratic algebra $\mathcal{A}$, which leaves α, β, γ, δ, and A_1 fixed. An arbitrary linear combination of A_0, B_0, and C_0 may be achieved by the following formula:

$$F = \text{tr}(AX) = 2aA_0 + bB_0 + cC_0 \tag{A.6}$$

where

$$A = \begin{pmatrix} a & c \\ b & -a \end{pmatrix}. \tag{A.7}$$

Substituting $X(\theta)$ for X in (A.6) leads to

$$F = \text{tr}(A(\theta)X) \tag{A.8}$$

where

$$A(\theta) = V(\theta)AV(-\theta). \tag{A.9}$$

This equation thus gives the result of applying the automorphism (A.5) to the linear combination (A.6). It is not necessary to know the explicit form of this equation, but rather the two main properties

$$\det A(\theta) = \det A \qquad \text{tr}(A(\theta)\Omega) = \text{tr}(A\Omega). \tag{A.10}$$

The second relation follows from the fact that $V(\theta)$ commutes with Ω. Letting $A(\theta) = \tilde{A}$ we may write the above relations explicitly as

$$a^2 + bc = \tilde{a}^2 + \tilde{b}\tilde{c} \qquad 2\alpha a + \beta b + \gamma c = 2\alpha\tilde{a} + \beta\tilde{b} + \gamma\tilde{c}. \tag{A.11}$$

These relations give two equations to determine the new parameters $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ in terms of the old parameters a, b, and c, in agreement with the additional arbitrary parameter θ. We may thus choose to put any one of the new parameters to zero and thus determine the values of the remaining two.

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